Inverse Trigonometric Functions

1 Mark Questions

1. If $\sin\left(\sin^{-1}\dfrac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.

Delhi 2014

Given, $\sin\left(\sin^{-1}\dfrac{1}{5} + \cos^{-1}x\right) = 1$

$\Rightarrow \quad \sin^{-1}\dfrac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$

$$[\because \sin\theta = x \Rightarrow \theta = \sin^{-1}x]$$

$\Rightarrow \sin^{-1}\dfrac{1}{5} + \cos^{-1}x = \sin^{-1}\left(\sin\dfrac{\pi}{2}\right)\left[\because \sin\dfrac{\pi}{2} = 1\right]$

$\Rightarrow \quad \sin^{-1}\dfrac{1}{5} + \cos^{-1}x = \dfrac{\pi}{2}$ (1/2)

But we know that, $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$,

$x \in [-1, 1]$

$\therefore \quad \sin^{-1}\dfrac{1}{5} = \sin^{-1}x \Rightarrow x = \dfrac{1}{5}$ (1/2)

2. If $\tan^{-1} x + \tan^{-1} y = \dfrac{\pi}{4}$; $xy < 1$, then write the value of $x + y + xy$. **All India 2014**

Given, $\tan^{-1} x + \tan^{-1} y = \dfrac{\pi}{4}$, $xy < 1$

We know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\dfrac{x + y}{1 - xy}\right], \ xy < 1$$

$$\therefore \tan^{-1}\left[\dfrac{x + y}{1 - xy}\right] = \dfrac{\pi}{4} \ \Rightarrow \ \dfrac{x + y}{1 - xy} = \tan\dfrac{\pi}{4} \quad \textbf{(1/2)}$$

$$\Rightarrow \qquad x + y = 1 - xy \qquad \left[\because \tan\dfrac{\pi}{4} = 1\right]$$

$$\Rightarrow \qquad x + y + xy = 1 \qquad\qquad\qquad \textbf{(1/2)}$$

3. Write the value of $\cos^{-1}\left(-\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$.

Foreign 2014

We have, $\cos^{-1}\left(\dfrac{-1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$

$= \left[\pi - \cos^{-1}\left(\dfrac{1}{2}\right)\right] + 2\sin^{-1}\left(\dfrac{1}{2}\right)$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$

$= \left[\pi - \cos^{-1}\left(\cos\dfrac{\pi}{3}\right)\right] + 2\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$ **(1/2)**

$[\because$ principal value branch of $\cos^{-1}x$ is $[0, \pi]$

and that of $\sin^{-1}x$ is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]]$

$= \left[\pi - \dfrac{\pi}{3}\right] + 2 \times \dfrac{\pi}{6}$

$= \dfrac{2\pi}{3} + \dfrac{2\pi}{6} = \dfrac{4\pi + 2\pi}{6} = \pi$ **(1/2)**

4. Write the principal value of $\cos^{-1}[\cos(680)°]$.

Delhi 2014C

> ? Firstly, write $680°$ in the form of $(2\pi - \theta)$ and then use $\cos(2\pi - \theta) = \cos\theta$ and $\cos^{-1}(\cos\theta) = \theta$.

We have, $\cos^{-1}[\cos(680)°]$

$= \cos^{-1}[\cos(2 \times 360 - 40)°]$

$= \cos^{-1}[\cos 40°]$ $\qquad [\because \cos(2\pi - \theta) = \cos\theta]$

$= 40°$ $\qquad [\because \cos^{-1}(\cos\theta) = \theta]$ **(1)**

5. Write the principal value of $\tan^{-1}\left[\sin\left(\dfrac{-\pi}{2}\right)\right]$.

All India 2014C

We have, $\tan^{-1}\left[\sin\left(-\dfrac{\pi}{2}\right)\right]$

$$= \tan^{-1}[-1] \qquad \left[\because \sin\left(\dfrac{-\pi}{2}\right) = -1\right]$$

$\because$ Principal value branch of $\tan^{-1}$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

$$\therefore \tan^{-1}(-1) = \tan^{-1}\left(-\tan\dfrac{\pi}{4}\right) \qquad \left[\because \tan\dfrac{\pi}{4} = 1\right]$$

$$= \tan^{-1}\left[\tan\left(\dfrac{-\pi}{4}\right)\right] = \dfrac{-\pi}{4} \quad \textbf{(1)}\ [\ \tan^{-1}(\tan\theta) = \theta]$$

6. Find the value of the following:

$$\cot\left(\dfrac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right) \qquad \qquad \text{All India 2014 C}$$

> ❓ Firstly, use the property $\cot\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta$, then put $\cot^{-1}\sqrt{3} = \dfrac{\pi}{6}$ and simplify.

We have, $\cot\left[\dfrac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right]$

$$= \tan(2\cot^{-1}\sqrt{3})\left[\because \cot\left(\dfrac{\pi}{2} - \theta\right) = \tan\theta\right] \quad \textbf{(1/2)}$$

$$= \tan\left(2 \times \dfrac{\pi}{6}\right)\left[\because \cot^{-1}\sqrt{3} = \cot^{-1}\left(\cot\dfrac{\pi}{6}\right) = \dfrac{\pi}{6}\right]$$

$$= \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3} \qquad \qquad \textbf{(1/2)}$$

7. Write the principal value of

$$\left[\cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\left(-\dfrac{1}{2}\right)\right].$$

$$\text{Delhi 2013C}$$

We have, $\cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\left(-\dfrac{1}{2}\right)$

$= \cos^{-1}\dfrac{\sqrt{3}}{2} + \left[\pi - \cos^{-1}\left(\dfrac{1}{2}\right)\right]$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1}x]$

$= \dfrac{\pi}{6} + \pi - \dfrac{\pi}{3} = \dfrac{\pi + 6\pi - 2\pi}{6} = \dfrac{5\pi}{6}$ **(1)**

8. Write the value of $\tan^{-1}\left(\dfrac{a}{b}\right) - \tan^{-1}\left(\dfrac{a-b}{a+b}\right)$.

<div align="right">Delhi 2013C</div>

$\tan^{-1}\left(\dfrac{a}{b}\right) - \tan^{-1}\left(\dfrac{a-b}{a+b}\right) = \tan^{-1}\dfrac{\left[\dfrac{a}{b} - \left(\dfrac{a-b}{a+b}\right)\right]}{1 + \dfrac{a}{b}\left(\dfrac{a-b}{a+b}\right)}$

$\left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right)\right]$

$= \tan^{-1}\left(\dfrac{a^2 + ab - ab + b^2}{ab + b^2 + a^2 - ab}\right)$ **(1/2)**

$= \tan^{-1}\left(\dfrac{a^2 + b^2}{a^2 + b^2}\right) = \tan^{-1}1$

$= \tan^{-1}\left(\tan\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ $[\because \tan^{-1}(\tan\theta) = \theta]$ **(1/2)**

9. Write the principal value of

$\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right)$.   HOTS; Delhi 2013

> **?** Firstly, we check the given angle in principal value branch. If it is not so, then convert it. After that, use the identity
> $$\tan^{-1}(\tan\theta) = \theta, \cos^{-1}(\cos\theta) = \theta.$$

$\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right)$

$$= \tan^{-1}\left(\tan\dfrac{\pi}{4}\right) + \cos^{-1}\left(-\cos\dfrac{\pi}{3}\right) \quad (1/2)$$

$$= \dfrac{\pi}{4} + \cos^{-1}\left[\cos\left(\pi - \dfrac{\pi}{3}\right)\right]$$

[since, principal value branch of $\cos^{-1}x$ is $[0, \pi]$, so we convert $-\cos\theta = \cos(\pi - \theta)$]

$$= \dfrac{\pi}{4} + \cos^{-1}\left[\cos\dfrac{2\pi}{3}\right]$$

$$= \dfrac{\pi}{4} + \dfrac{2\pi}{3} = \dfrac{3\pi + 8\pi}{12}$$

$$= \dfrac{11\pi}{12} \qquad (1/2)$$

10. Write the value of $\tan\left(2\tan^{-1}\dfrac{1}{5}\right)$. **Delhi 2013**

$$\tan\left(2\tan^{-1}\dfrac{1}{5}\right) = \tan\left[\tan^{-1}\left(\dfrac{2\times\dfrac{1}{5}}{1-\left(\dfrac{1}{5}\right)^2}\right)\right] \quad (1/2)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\dfrac{2\times 5}{24}\right)\right] = \tan\left[\tan^{-1}\left(\dfrac{5}{12}\right)\right] = \dfrac{5}{12}$$

$$[\because \tan(\tan^{-1}\theta) = \theta] \quad (1/2)$$

11. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\dfrac{\sqrt{3}}{2}\right)\right]$.

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\dfrac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(2\cdot\dfrac{3}{4}-1\right)\right\}\right]$$

$$[\because 2\cos^{-1}x = \cos^{-1}(2x^2-1)]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\dfrac{3}{2}-1\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\left(\dfrac{1}{2}\right)\right\}\right] \qquad \textbf{(1/2)}$$

$$= \tan^{-1}\left[2\sin\left\{\cos^{-1}\cdot\left(\cos\dfrac{\pi}{3}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\dfrac{\pi}{3}\right][\because \cos^{-1}(\cos\theta)=\theta]$$

$$= \tan^{-1}\left(2\cdot\dfrac{\sqrt{3}}{2}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \tan^{-1}\left(\tan\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$$

$$[\because \tan^{-1}(\tan\theta)=\theta] \ \textbf{(1/2)}$$

12. Write the principal value of

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}).$$

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$$

$$= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\}$$

$$\left[\begin{array}{l} \because \text{principal value branch of } \cot^{-1}x \text{ is } (0, \pi.) \\ \therefore \cot^{-1}(-x) = \pi - \cot^{-1}x \end{array} \right]$$

$$= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3}$$

$$= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2} \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \right] \text{(1)}$$

13. Write the value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$.

<div align="right">Delhi 2012</div>

> Firstly, we check the given angle in principal value branch, then use the identity
> $$\sin^{-1}(\sin\theta) = \theta \text{ and } \cos(\cos^{-1}\theta) = \theta.$$

$$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \cos^{-1}\left(\cos\frac{\pi}{3}\right) - 2\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$$\left[\because \text{principal value branch of } \cos^{-1}x \text{ is } [0, \pi] \text{ and that of } \sin^{-1} \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right]$$

$$= \frac{\pi}{3} - 2\left(-\frac{\pi}{6}\right)$$

$$[\because \cos^{-1}(\cos\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3} \qquad \text{(1)}$$

14. Find the principal value of

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2).$$

> 💡 Given expression is not standard identity, so we separately find the value of $\tan^{-1}(\sqrt{3})$ and $\sec^{-1}(-2)$, then simplify it.

We know that, the principal value branch of $\tan^{-1}x$ is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and that of $\sec^{-1}$ is $[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$.

$$\therefore \tan^{-1}\sqrt{3} - \sec^{-1}(-2)$$

$$= \tan^{-1}\left(\tan\dfrac{\pi}{3}\right) - \sec^{-1}\left(\sec\dfrac{2\pi}{3}\right)$$

$$\left[\because \tan\dfrac{\pi}{3} = \sqrt{3} \text{ and } \sec\dfrac{2\pi}{3} = -2\right]$$

$$= \dfrac{\pi}{3} - \dfrac{2\pi}{3} = \dfrac{-\pi}{3}$$

$[\because \tan^{-1}(\tan\theta) = \theta \text{ and } \sec^{-1}(\sec\theta) = \theta]$ **(1)**

15. Using the principal values, write the value of $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$.

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and of $\sin^{-1} x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$$\therefore \quad \cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$$

$$= \cos^{-1}\left(\cos\dfrac{\pi}{3}\right) + 2\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$$

$$\left[\because \cos\dfrac{\pi}{3} = \dfrac{1}{2} \text{ and } \sin\dfrac{\pi}{6} = \dfrac{1}{2}\right]$$

$$= \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$$

$$[\because \cos^{-1}(\cos\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]\ \textbf{(1)}$$

16. Write the value of $\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right]$.

> Firstly, find the principal value of $\sin^{-1}\left(\dfrac{1}{2}\right)$, then solve it.

$$\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right] = \sin\left[\dfrac{\pi}{3} + \sin^{-1}\left(\dfrac{1}{2}\right)\right]$$

$$[\because \sin^{-1}(-\theta) = -\sin^{-1}\theta]$$

$$= \sin\left[\dfrac{\pi}{3} + \sin^{-1}\left(\sin\dfrac{\pi}{6}\right)\right]\left[\because \sin\dfrac{\pi}{6} = \dfrac{1}{2}\right]$$

$$= \sin\left[\dfrac{\pi}{3} + \dfrac{\pi}{6}\right] = \sin\dfrac{\pi}{2} = 1$$

$$[\because \sin^{-1}(\sin\theta) = \theta]\ \textbf{(1)}$$

NOTE Please be careful that we do not write $\sin^{-1}(-\sin\theta) = \theta$.

17. Write the value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

> Firstly, we check the given angle in principal value. If it is so, then use the identity $\tan^{-1}(\tan\theta)=\theta$.

We know that, principal value branch of $\tan^{-1}x$ is $\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$.

$\therefore$ Principal value of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

$= \tan^{-1}\left[\tan\left(\pi-\dfrac{\pi}{4}\right)\right]$

$$\left[\because \dfrac{3\pi}{4}\notin\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)\right.$$
$$\left.\text{so, write }\dfrac{3\pi}{4}\text{ as}\left(\pi-\dfrac{\pi}{4}\right)\right]$$

$= \tan^{-1}\left(-\tan\dfrac{\pi}{4}\right)\qquad [\because \tan(\pi-\theta)=-\tan\theta]$

$= \tan^{-1}\left[\tan\left(-\dfrac{\pi}{4}\right)\right]\qquad [\because (-\tan\theta)=\tan(-\theta)]$

$= -\dfrac{\pi}{4}\in\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$

$\therefore\qquad \tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)=-\dfrac{\pi}{4}\qquad\qquad (1)$

NOTE Please be careful, we do not write $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)=\dfrac{3\pi}{4}$, because $\dfrac{3\pi}{4}\notin\left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right)$.

18. Write the value of $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$.

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$$

$$\left[\begin{array}{l} \because \dfrac{7\pi}{6} \notin [0, \pi], \text{ so} \\[2mm] \text{write}\dfrac{7\pi}{6} = \left(2\pi - \dfrac{5\pi}{6}\right) \end{array}\right]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \quad \left[\begin{array}{l} \because \cos(2\pi - \theta) = \cos\theta \\[2mm] \text{and } \dfrac{5\pi}{6} \in [0, \pi] \end{array}\right]$$

$$= \frac{5\pi}{6}$$

$$\therefore \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6} \qquad\qquad (1)$$

19. What is the principal value of
$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)?$$

All India 2011, 2008, 2009C

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and for $\sin^{-1} x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$$\therefore \cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) + \sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$$

$$= \dfrac{2\pi}{3} + \sin^{-1}\left[\sin\left(\pi - \dfrac{\pi}{3}\right)\right]$$

$$\left[\begin{array}{l} \because \dfrac{2\pi}{3} \notin \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right], \text{ so} \\ \text{write } \dfrac{2\pi}{3} \text{ as } \left(\pi - \dfrac{\pi}{3}\right) \end{array}\right]$$

$$= \dfrac{2\pi}{3} + \sin^{-1}\left(\sin\dfrac{\pi}{3}\right) \quad [\because \sin(\pi - \theta) = \sin\theta]$$

$$= \dfrac{2\pi}{3} + \dfrac{\pi}{3} = \dfrac{3\pi}{3} = \pi \quad [\because \sin^{-1}(\sin\theta) = \theta] \quad \textbf{(1)}$$

20. What is the principal value of $\tan^{-1}(-1)$?

We know that, the principal value branch of $\tan^{-1} x$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

$$\therefore \tan^{-1}(-1) = \tan^{-1}\left(-\tan\dfrac{\pi}{4}\right) \quad \left[\because \tan\dfrac{\pi}{4} = 1\right]$$

$$= \tan^{-1}\left[\tan\left(-\dfrac{\pi}{4}\right)\right] \quad [\because (-\tan\theta) = \tan(-\theta)]$$

$$= -\dfrac{\pi}{4} \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$$

Hence, $\tan^{-1}(-1) = -\dfrac{\pi}{4}$ \hfill **(1)**

21. Using the principal values, write the value of $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$.

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$.

$$\therefore \quad \sin^{-1} \left(-\dfrac{\sqrt{3}}{2} \right) = \sin^{-1} \left(-\sin \dfrac{\pi}{3} \right)$$

$$= \sin^{-1} \left[\sin \left(-\dfrac{\pi}{3} \right) \right] \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\dfrac{\pi}{3} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right] \qquad [\because \sin^{-1}(\sin\theta) = \theta]$$

Hence, $\quad \sin^{-1} \left(-\dfrac{\sqrt{3}}{2} \right) = -\dfrac{\pi}{3}$ \hfill **(1)**

22. Write the principal value of $\sin^{-1} \left(-\dfrac{1}{2} \right)$.

Delhi 2011C

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$.

$$\therefore \quad \sin^{-1} \left(-\dfrac{1}{2} \right) = \sin^{-1} \left(-\sin \dfrac{\pi}{6} \right)$$

$$= \sin^{-1} \left[\sin \left(-\dfrac{\pi}{6} \right) \right] \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\dfrac{\pi}{6} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right] \qquad [\because \sin^{-1}(\sin\theta) = \theta]$$

Hence, $\quad \sin^{-1} \left(-\dfrac{1}{2} \right) = -\dfrac{\pi}{6}$ \hfill **(1)**

23. Write the principal value of $\sin^{-1} \left(\dfrac{\sqrt{3}}{2} \right)$.

Delhi 2010

We know that, the principal value branch of

$$\sin^{-1} x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

$$\therefore \quad \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \sin^{-1} \left(\sin \frac{\pi}{3} \right) \left[\because \ \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \right]$$

$$= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$[\because \sin^{-1} (\sin \theta) = \theta] \ \textbf{(1)}$$

24. What is the principal value of $\sec^{-1} (-2)$?

We know that, the principal value branch of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$.

$$\therefore \sec^{-1} (-2) = \sec^{-1} \left(-\sec \frac{\pi}{3} \right)$$

$$\left[\text{here, } \sec^{-1}(-\sec \frac{\pi}{3}) \neq -\frac{\pi}{3} \right.$$

$$\left. \text{since, } \frac{-\pi}{3} \notin [0, \pi] - \left\{ \frac{\pi}{2} \right\} \right]$$

$$= \sec^{-1} \left[\sec \left(\pi - \frac{\pi}{3} \right) \right] \ [\because \ \sec (\pi - \theta) = -\sec \theta]$$

$$= \sec^{-1} \left(\sec \frac{2\pi}{3} \right) = \frac{2\pi}{3} \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

$$[\because \sec^{-1} (\sec \theta) = \theta]$$

Hence, $\sec^{-1} (-2) = \dfrac{2\pi}{3}$ \hfill **(1)**

25. What is the domain of the function $\sin^{-1} x$?

The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$. \hfill **(1)**

26. Using the principal values, find the value of $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$.

As the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\therefore \quad \cos^{-1}\left(\cos\dfrac{13\pi}{6}\right) \neq \dfrac{13\pi}{6} \text{ as } \dfrac{13\pi}{6} \notin [0, \pi]$$

Now, $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \dfrac{\pi}{6}\right)\right]$

$$= \cos^{-1}\left(\cos\dfrac{\pi}{6}\right) \quad [\because \cos(2\pi + \theta) = \cos\theta]$$

$$= \dfrac{\pi}{6} \in [0, \pi] \qquad [\because \cos^{-1}(\cos\theta) = \theta]$$

Hence, $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right) = \dfrac{\pi}{6}$ **(1)**

27. If $\tan^{-1}(\sqrt{3}) + \cot^{-1} x = \dfrac{\pi}{2}$, then find the value of x.

Given, $\tan^{-1}\sqrt{3} + \cot^{-1} x = \dfrac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{3} = \dfrac{\pi}{2} - \cot^{-1} x$

$\Rightarrow \tan^{-1}\sqrt{3} = \tan^{-1} x \left[\because \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}\right]$

On equating both sides, we get

$$x = \sqrt{3} \qquad \qquad \text{(1)}$$

28. Write the principal value of $\sin^{-1}\left(\sin\dfrac{3\pi}{5}\right)$.

As we know that, the principal value branch of $\sin^{-1} x$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$\therefore \ \sin^{-1}\left[\sin\left(\dfrac{3\pi}{5}\right)\right] \neq \dfrac{3\pi}{5} \quad \left[\because \ \dfrac{3\pi}{5} \notin \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right]$

Now, $\sin^{-1}\left(\sin\dfrac{3\pi}{5}\right)$

$\qquad = \sin^{-1}\left[\sin\left(\pi - \dfrac{2\pi}{5}\right)\right]$

$\qquad = \sin^{-1}\left(\sin\dfrac{2\pi}{5}\right) \ [\because \ \sin(\pi - \theta) = \sin\theta]$

$\qquad = \dfrac{2\pi}{5} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin\theta) = \theta]$

Hence, $\sin^{-1}\left(\sin\dfrac{3\pi}{5}\right) = \dfrac{2\pi}{5}$ \hfill (1)

29. Using the principal values, evaluate
$\tan^{-1}(1) + \sin^{-1}\left(-\dfrac{1}{2}\right)$.

<div align="right">Delhi 2009C</div>

We know that, the principal value of $\tan^{-1} x$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and that of $\sin^{-1}$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$\therefore \tan^{-1}(1) + \sin^{-1}\left(-\dfrac{1}{2}\right)$

$= \tan^{-1}\left(\tan \dfrac{\pi}{4}\right) + \sin^{-1}\left(-\sin \dfrac{\pi}{6}\right)$

$$\left[\because \ \tan \dfrac{\pi}{4} = 1 \text{ and } \sin \dfrac{\pi}{6} = \dfrac{1}{2}\right]$$

$= \tan^{-1}\left(\tan \dfrac{\pi}{4}\right) + \sin^{-1}\left[\sin\left(-\dfrac{\pi}{6}\right)\right]$

$$[\because \sin(-\theta) = -\sin\theta]$$

$= \dfrac{\pi}{4} - \dfrac{\pi}{6} = \dfrac{\pi}{12}$

$[\because \tan^{-1}(\tan\theta) = \theta \text{ and } \sin^{-1}(\sin\theta) = \theta]$ **(1)**

30. Find the principal value of $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.

<div align="right">All India 2008C</div>

$\cos^{-1}\dfrac{\sqrt{3}}{2} = \cos^{-1}\left(\cos \dfrac{\pi}{6}\right)$ $\qquad \left[\because \cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}\right]$

Since, $\dfrac{\pi}{6} \in [0, \pi]$

{As principal value branch of $\cos^{-1} x$ is $[0, \pi]$}

$\therefore$

$\cos^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6}$ $\hspace{4cm}$ **(1)**

6 Mark Questions

31. Solve the following equation:

$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\dfrac{3}{4}\right)$

<div align="right">Foreign 2014; All India 2013; HOTS</div>

Given, $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \dfrac{3}{4}\right)$

$\Rightarrow \quad \sin\left\{\dfrac{\pi}{2} - \tan^{-1} x\right\} = \sin\left(\cot^{-1} \dfrac{3}{4}\right)$

$$\left[\because \sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta\right] \textbf{(1)}$$

On equating both sides, we get

$$\dfrac{\pi}{2} - \tan^{-1} x = \cot^{-1} \dfrac{3}{4} \qquad\qquad \textbf{(1)}$$

$\Rightarrow \qquad \tan^{-1} x + \cot^{-1} \dfrac{3}{4} = \dfrac{\pi}{2} \qquad\qquad \textbf{(1)}$

It is only possible, when $x = \dfrac{3}{4}$.

$$\left[\because \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}, x \in R\right] \textbf{(1)}$$

32. Solve following equation for x.

$$\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1} x$$

Foreign 2011C, 08C; All India 2014C, 2010, 2009C

Multiply both sides by 2 and then in LHS, use the relation $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ and solve it. Then, equate both sides to get the value of x.

Given, $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1}x$, $x > 0$

$\Rightarrow \quad 2\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \tan^{-1}x$

$\Rightarrow \quad \tan^{-1}\left[\dfrac{2\left(\dfrac{1-x}{1+x}\right)}{1-\left(\dfrac{1-x}{1+x}\right)^2}\right] = \tan^{-1}x$ $\qquad$ (1½)

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right]$$

$\Rightarrow \quad \tan^{-1}\left[\dfrac{2(1-x)(1+x)}{(1+x)^2 - (1-x)^2}\right] = \tan^{-1}x$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{2-2x^2}{4x}\right) = \tan^{-1}x$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

$\Rightarrow \quad \dfrac{1-x^2}{2x} = x \quad \Rightarrow \quad 1-x^2 = 2x^2$

$\Rightarrow \quad 3x^2 = 1 \Rightarrow x^2 = \dfrac{1}{3} \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$ (1½)

But given, $x > 0$

$\therefore \qquad x = \dfrac{1}{\sqrt{3}}$ $\qquad\qquad$ (1)

33. Solve for x, $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$.

Given equation is

$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

$\Rightarrow \tan^{-1}\left(\dfrac{2 \cos x}{1 - \cos^2 x}\right) = \tan^{-1}\left(\dfrac{2}{\sin x}\right)$

$\left[\because 2 \tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)\right]$ **(1½)**

$\Rightarrow \dfrac{2 \cos x}{\sin^2 x} = \dfrac{2}{\sin x}$ $[\because \sin^2 x = 1 - \cos^2 x] \dots$(i)

$\Rightarrow \sin x \cdot \cos x - \sin^2 x = 0$

$\Rightarrow \sin x (\cos x - \sin x) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = \sin x$

$\Rightarrow \sin x = \sin 0$ or $\cot x = 1 = \cot \pi/4$

$\Rightarrow \qquad\qquad x = 0$ or $\dfrac{\pi}{4}$ **(1½)**

But here at $x = 0$, the given equation does not exist. Hence, $x = \dfrac{\pi}{4}$ is the only solution. **(1)**

34. Solve for x, $\tan^{-1} x + 2 \cot^{-1} x = \dfrac{2\pi}{3}$.

The given equation is $\tan^{-1} x + 2 \cot^{-1} x = \dfrac{2\pi}{3}$

We know that, $\cot^{-1} x = \tan^{-1}\dfrac{1}{x}$

So, the given equation can be written as

$\tan^{-1} x + 2 \tan^{-1}\left(\dfrac{1}{x}\right) = \dfrac{2\pi}{3}$

$\Rightarrow \tan^{-1} x + \tan^{-1}\left(\dfrac{2 \times \dfrac{1}{x}}{1 - \dfrac{1}{x^2}}\right) = \dfrac{2\pi}{3}$

$$\left(1 - \dfrac{}{x^2} \right)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2} \right) \right]$$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\left(\dfrac{\dfrac{2}{x}}{\dfrac{x^2-1}{x^2}} \right) = \dfrac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{x^2-1} \right) = \dfrac{2\pi}{3} \qquad \text{(1½)}$$

$$\Rightarrow \quad \tan^{-1}\left(\dfrac{x + \dfrac{2x}{x^2-1}}{1 - \dfrac{2x^2}{x^2-1}} \right) = \dfrac{2\pi}{3}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy} \right) \right]$$

$$\Rightarrow \quad \dfrac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan\dfrac{2\pi}{3} \qquad \text{(1½)}$$

$$\Rightarrow \quad \dfrac{x^3 + x}{-1 - x^2} = \tan\left(\pi - \dfrac{\pi}{3} \right)$$

$$\Rightarrow \quad \dfrac{x^3 + x}{-(1 + x^2)} = -\tan\dfrac{\pi}{3}$$

$$[\because \ \tan(\pi - \theta) = -\tan\theta]$$

$$\Rightarrow \quad \dfrac{x(1 + x^2)}{-(1 + x^2)} = -\sqrt{3} \ \Rightarrow x = \sqrt{3} \qquad \text{(1)}$$

35. Prove that
$$\cot^{-1}\left(\dfrac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right) = \dfrac{x}{2}; x \in \left(0, \dfrac{\pi}{4} \right).$$

Delhi 2014C, 2011; All India 2009

$$\bullet \; \text{LHS} = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right.$$

$$\left.\times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}\right] \quad \textbf{(1)}$$

[multiplying and dividing by conjugate of denominator, i.e. by $\sqrt{1+\sin x} + \sqrt{1-\sin x}$]

$$= \cot^{-1}\left[\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2}\right] \quad \textbf{(1)}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \cot^{-1}\left[\frac{1+\sin x + 1 - \sin x + 2\sqrt{1-\sin^2 x}}{1 + \sin x - 1 + \sin x}\right]$$

$$= \cot^{-1}\left[\frac{2+2\cos x}{2\sin x}\right] \quad [\because 1-\sin^2 x = \cos^2 x] \;\textbf{(1)}$$

$$= \cot^{-1}\left[\frac{1+\cos x}{\sin x}\right] = \cot^{-1}\left[\frac{2\cos^2 \dfrac{x}{2}}{2\sin \dfrac{x}{2}\cos \dfrac{x}{2}}\right]$$

$$\left[\because \cos x = 2\cos^2 \frac{x}{2} - 1 \text{ and } \sin x = 2\sin \frac{x}{2}\cos \frac{x}{2}\right]$$

$$= \cot^{-1}\left[\cot \frac{x}{2}\right] = \frac{x}{2} = \text{RHS} \quad \textbf{(1)}$$

Hence proved.

36. Prove that

$$2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Delhi 2014

. LHS $= 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$

$= 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$

$= 2\tan^{-1}\left\{\dfrac{\dfrac{1}{5} + \dfrac{1}{8}}{1 - \dfrac{1}{5} \times \dfrac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\dfrac{5\sqrt{2}}{7}\right)^2 - 1}$

$$\left[\begin{array}{l} \because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left\{\dfrac{x+y}{1-xy}\right\} \\ \text{and } \sec^{-1}x = \tan^{-1}\sqrt{x^2 - 1} \end{array}\right]$$ **(1)**

$= 2\tan^{-1}\dfrac{13}{39} + \tan^{-1}\sqrt{\dfrac{50}{49} - 1}$

$= 2\tan^{-1}\dfrac{i}{3} + \tan\quad\quad 7$ (i)

$$= \tan^{-1}\left\{\frac{2 \times \frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\} + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left\{\frac{2x}{1-x^2}\right\}\right] \quad (1)$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \qquad (1)$$

$$[\because \tan^{-1}(\tan\theta) = \theta] \text{ \textbf{Hence proved.}}$$

37. Prove that $\tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, \ \frac{-1}{\sqrt{2}} \le x \le 1.$$

All India 2014, 2011, 2014C; HOTS

$$\text{LHS} = \tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$$

Let $x = \cos\theta$, so that $\cos^{-1}x = \theta$ **(1)**

Then, $\text{LHS} = \tan^{-1}\left[\dfrac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}\right]$

$$= \tan^{-1}\left[\dfrac{\sqrt{2}\,\cos\dfrac{\theta}{2} - \sqrt{2}\,\sin\dfrac{\theta}{2}}{\sqrt{2}\,\cos\dfrac{\theta}{2} + \sqrt{2}\,\sin\dfrac{\theta}{2}}\right]$$

$$\left[\because 1 + \cos\theta = 2\cos^2\dfrac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}\right]$$

                                (1)

$$= \tan^{-1}\left[\dfrac{\cos\dfrac{\theta}{2} - \sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2}}\right] = \tan^{-1}\left[\dfrac{1 - \tan\dfrac{\theta}{2}}{1 + \tan\dfrac{\theta}{2}}\right]$$

[dividing numerator and denominator by
$\cos(\theta/2)$] **(1)**

$$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)\right]$$

$$\left[\because \tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x\, \tan y}\right]$$

$$= \dfrac{\pi}{4} - \dfrac{\theta}{2} \qquad\qquad [\because \tan^{-1}(\tan\theta) = \theta]$$

$$= \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x \qquad\qquad [\because \theta = \cos^{-1}x]$$

$= \text{RHS}$ **Hence proved.**

38. If $\tan^{-1}\left(\dfrac{x-2}{x-4}\right) + \tan^{-1}\left(\dfrac{x+2}{x+4}\right) = \dfrac{\pi}{4}$, then
find the value of x.   **All India 2014**

Given, $\tan^{-1}\left(\dfrac{x-2}{x-4}\right) + \tan^{-1}\left(\dfrac{x+2}{x+4}\right) = \dfrac{\pi}{4}$

$\Rightarrow \tan^{-1}\left[\dfrac{\dfrac{x-2}{x-4} + \dfrac{x+2}{x+4}}{1 - \left(\dfrac{x-2}{x-4}\right)\left(\dfrac{x+2}{x+4}\right)}\right] = \dfrac{\pi}{4}$

$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right]$ **(1)**

$\Rightarrow \tan^{-1}\left[\dfrac{\dfrac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)}}{\dfrac{(x-4)(x+4)-(x-2)(x+2)}{(x-4)(x+4)}}\right] = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{x^2 - 2x + 4x - 8 + x^2 - 4x + 2x - 8}{(x^2 - 16) - (x^2 - 4)}$

$\hspace{5cm} = \tan\dfrac{\pi}{4}$ **(1)**

$\Rightarrow \hspace{2cm} \dfrac{2x^2 - 16}{-12} = 1$

$\Rightarrow \hspace{2cm} 2x^2 - 16 = -12$

$\Rightarrow \hspace{2cm} 2x^2 = -12 + 16$

$\Rightarrow \hspace{2cm} 2x^2 = 4 \Rightarrow x^2 = 2$

$\therefore \hspace{2.5cm} x = \pm\sqrt{2}$ **(1)**

Hence, $\sqrt{2}$ and $-\sqrt{2}$ are the required values of x.

(1)

39. Prove that

$\cos^{-1}(x) + \cos^{-1}\left\{\dfrac{x}{2} + \dfrac{\sqrt{3-3x^2}}{2}\right\} = \dfrac{\pi}{3}.$

All India 2014C

We have to prove

$$\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3 - 3x^2}}{2}\right\} = \frac{\pi}{3}$$

$$\text{LHS} = \cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3 - 3x^2}}{2}\right\}$$

Let $\qquad \cos^{-1}x = \alpha \implies x = \cos\alpha \qquad$ **(1)**

Then, $\qquad \text{LHS} = \alpha + \cos^{-1}$

$$\left[\cos\alpha \cdot \cos\frac{\pi}{3} + \frac{\sqrt{3}}{2}\sqrt{1 - \cos^2\alpha}\right]$$

$$= \alpha + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right]$$

$$\left[\because \sin\alpha = \sqrt{1 - \cos^2\alpha},\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right] \textbf{(1)}$$

$$= \alpha + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \alpha\right)\right]$$

$$[\because \cos(A - B) = \cos A \cos B + \sin A \sin B]\ \textbf{(1)}$$

$$= \alpha + \frac{\pi}{3} - \alpha = \frac{\pi}{3} = \text{RHS} \qquad \textbf{(1)}$$

40. Prove that $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$

Foreign 2014

To prove, $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$

$\text{LHS} = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18}$$

$$\left[\because \cot^{-1}x = \frac{1}{\tan^{-1}x} \right] \quad (1)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \left(\frac{1}{8} \right)} \right) + \tan^{-1}\frac{1}{18}$$

$$\left[\because \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB} \right) \right] \quad (1)$$

$$= \tan^{-1}\left(\frac{15}{55} \right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{3}{11} \right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \left(\frac{3}{11} \right) \times \left(\frac{1}{18} \right)} \right) = \tan^{-1}\left(\frac{65}{195} \right) \quad (1)$$

$$= \tan^{-1}\left(\frac{1}{3} \right) = \cot^{-1}3 = \text{RHS} \quad (1)$$

Hence proved.

41. Prove the following:

$$\cot^{-1}\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right] = \frac{x}{2} \; ; x \in \left(0, \frac{\pi}{4} \right)$$

Delhi 2014, 2011; HOTS; All India 2009

Use the relation

$$1 + \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2$$

For $x \in \left(0, \frac{\pi}{4} \right)$,

$$1 - \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2 \text{ and then simplify.}$$

$$\text{LHS} = \cot^{-1} \left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right)$$

$$= \cot^{-1} \left[\frac{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} + \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}}{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} - \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}} \right]$$

(1½)

$$\text{LHS} = \cot^{-1} \left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2} + \sin \frac{x}{2}} \right) \text{(1)}$$

$$= \cot^{-1} \left(\frac{2 \cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right) = \cot^{-1} \left(\cot \frac{x}{2} \right) \quad \text{(1/2)}$$

$\because$ The principal value branch of $\cot^{-1} x$ is $(0, \pi)$.

$$\therefore \qquad \cot^{-1} \left(\cot \frac{x}{2} \right) = \frac{x}{2}$$

$\therefore$ LHS = RHS   **Hence proved.**

NOTE If $x \in \left(0, \dfrac{\pi}{2}\right)$, then $\sqrt{1 - \sin x} = \cos \dfrac{x}{2} - \sin \dfrac{x}{2}$

and if $x \in \left(\dfrac{\pi}{2}, \pi\right)$, then $\sqrt{1 - \sin x} = \sin \dfrac{x}{2} - \cos \dfrac{x}{2}$

Alternate Method

$$\text{LHS} = \cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right.$$

$$\left. \times \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}\right)$$

[by rationalising denominator] **(1)**

$$= \cot^{-1}\left[\frac{(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})^2}{(\sqrt{1 + \sin x})^2 - (\sqrt{1 - \sin x})^2}\right]$$

$$[\because (a + b)(a - b) = a^2 - b^2]$$

$$= \cot^{-1}\left(\frac{1 + \sin x + 1 - \sin x + 2\sqrt{1 - \sin^2 x}}{1 + \sin x - 1 + \sin x}\right).$$

$$[\because (a + b)^2 = a^2 + b^2 + 2ab]$$

$$= \cot^{-1}\left(\frac{2 + 2\cos x}{2\sin x}\right) [\because \cos x = \sqrt{1 - \sin^2 x}] \ (1)$$

$$= \cot^{-1}\left(\frac{1 + \cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}}\right) \quad (1)$$

$$\left[\because 1 + \cos\theta = 2\cos^2 \frac{\theta}{2} \text{ and } \sin\theta = 2\sin \frac{\theta}{2}\cos \frac{\theta}{2}\right]$$

$$= \cot^{-1}\left[\cot \frac{x}{2}\right] = \frac{x}{2} = \text{RHS}$$

$$[\because \cot^{-1}(\cot\theta) = \theta] \ (1)$$

42. Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

All India 2014C; Delhi 2012, 2010C

To prove, $\sin^{-1}\left(\dfrac{8}{17}\right) + \sin^{-1}\left(\dfrac{3}{5}\right) = \cos^{-1}\left(\dfrac{36}{85}\right)$

Let $\sin^{-1}\left(\dfrac{8}{17}\right) = x$ ...(i)

and $\sin^{-1}\left(\dfrac{3}{5}\right) = y$ ...(ii)

$\Rightarrow \quad \sin x = \dfrac{8}{17} \quad$ and $\quad \sin y = \dfrac{3}{5}$ **(1)**

Now, $\cos^2 x = 1 - \sin^2 x$

$= 1 - \dfrac{64}{289} = \dfrac{225}{289} \Rightarrow \cos x = \sqrt{\dfrac{225}{289}}$

$\Rightarrow \quad \cos x = \dfrac{15}{17}$ **(1)**

Also, $\cos^2 y = 1 - \sin^2 y = 1 - \dfrac{9}{25}$

$\Rightarrow \quad \cos y = \sqrt{\dfrac{16}{25}} \quad \Rightarrow \cos y = \dfrac{4}{5}$ **(1)**

We know that,

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\Rightarrow \quad \cos(x + y) = \left(\dfrac{15}{17} \times \dfrac{4}{5}\right) - \left(\dfrac{8}{17} \times \dfrac{3}{5}\right)$

$\Rightarrow \quad \cos(x + y) = \dfrac{60}{85} - \dfrac{24}{85} = \dfrac{36}{85}$

$\Rightarrow \quad x + y = \cos^{-1}\left(\dfrac{36}{85}\right)$ ...(iii)

$\Rightarrow \quad \sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \cos^{-1}\dfrac{36}{85}$ **(1)**

[from Eqs. (i), (ii) and (iii)] **Hence proved.**

43. Show that $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right) = \dfrac{4 - \sqrt{7}}{3}$. *All India 2013*

. To prove, $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right) = \dfrac{4 - \sqrt{7}}{3}$

$\text{LHS} = \tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)$ ...(i)

$\dfrac{1}{2} \quad \dfrac{3}{4} \quad$ ...(ii)

Let $\quad\dfrac{1}{2}\sin^{-1}\left(\dfrac{3}{4}\right)=\theta \qquad\qquad\qquad ...(ii)$

$\Rightarrow\qquad \sin^{-1}\left(\dfrac{3}{4}\right)=2\theta \Rightarrow \sin 2\theta=\dfrac{3}{4} \qquad$ **(1)**

$\Rightarrow\qquad \dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{3}{4}\quad\left[\because \sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow\qquad\qquad 8\tan\theta=3+3\tan^2\theta$

$\Rightarrow 3\tan^2\theta-8\tan\theta+3=0$

Now, by Sridharacharya's rule

$$\tan\theta=\dfrac{8\pm\sqrt{64-36}}{2\times 3}=\dfrac{8\pm\sqrt{28}}{6} \qquad \textbf{(1)}$$

$$\Rightarrow=\dfrac{8\pm 2\sqrt{7}}{6}=\dfrac{4\pm\sqrt{7}}{3}\Rightarrow \theta=\tan^{-1}\left(\dfrac{4\pm\sqrt{7}}{3}\right)$$

$$[\because \tan\theta=\phi\Rightarrow\theta=\tan^{-1}\phi]$$

So, from Eq. (ii), we get

$$\dfrac{1}{2}\sin^{-1}\left(\dfrac{3}{4}\right)=\tan^{-1}\left(\dfrac{4\pm\sqrt{7}}{3}\right)\qquad\textbf{(1)}$$

Taking (−)ve sign, we get

$$\dfrac{1}{2}\sin^{-1}\left(\dfrac{3}{4}\right)=\tan^{-1}\left(\dfrac{4-\sqrt{7}}{3}\right)$$

On taking tan both sides, we get

$$\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\tan\left\{\tan^{-1}\left(\dfrac{4-\sqrt{7}}{3}\right)\right\}$$

$$\Rightarrow \tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$$

$$[\because \tan(\tan^{-1}\theta)=\theta]\ \textbf{(1)}$$

$=$ RHS $\qquad\qquad\qquad$ **Hence proved.**

44. Prove that $\sin^{-1}\dfrac{8}{17}+\sin^{-1}\dfrac{3}{5}=\tan^{-1}\dfrac{77}{36}$.

Delhi 2013C

To prove, $\sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{77}{36}$

Let $\sin^{-1}\dfrac{8}{17} = x$ and $\sin^{-1}\dfrac{3}{5} = y$

$\Rightarrow \quad \sin x = \dfrac{8}{17}$ and $\sin y = \dfrac{3}{5}$ $\qquad$ **(1)**

Now, $\cos^2 x = 1 - \sin^2 x = 1 - \dfrac{64}{289} = \dfrac{225}{289}$

$\Rightarrow \quad \cos x = \sqrt{\dfrac{225}{289}} = \dfrac{15}{17}$

Also, $\cos^2 y = 1 - \sin^2 y = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\Rightarrow \quad \cos y = \sqrt{16/25} = 4/5$ $\qquad$ **(1)**

We know that,
$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\qquad = \dfrac{15}{17} \times \dfrac{4}{5} - \dfrac{8}{17} \times \dfrac{3}{5} = \dfrac{60}{85} - \dfrac{24}{85} = \dfrac{36}{85}$

$\Rightarrow \quad x + y = \cos^{-1}\left(\dfrac{36}{85}\right)$

$\Rightarrow \quad \sin^{-1}\left(\dfrac{8}{17}\right) + \sin^{-1}\left(\dfrac{3}{5}\right) = \cos^{-1}\left(\dfrac{36}{85}\right)$ $\quad$ ...(i)

Again, let $\cos^{-1}\left(\dfrac{36}{85}\right) = z \Rightarrow \cos z = \dfrac{36}{85}$

then, $\tan z = \dfrac{77}{36}$ $\qquad$ **(1)**

$\Rightarrow \quad z = \tan^{-1}\dfrac{77}{36} \Rightarrow \cos^{-1}\dfrac{36}{85} = \tan^{-1}\dfrac{77}{36}$

On putting this value in Eq. (i), we get
$\sin^{-1}\left(\dfrac{8}{17}\right) + \sin^{-1}\left(\dfrac{3}{5}\right) = \tan^{-1}\left(\dfrac{77}{36}\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad$ **(1)**

Hence proved.

45. Solve for x, $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$.

$\qquad\qquad$ Delhi 2013C, 2009 ; All India 2009C, 2008

Given equation is $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\dfrac{3x + 2x}{1 - 3x \times 2x}\right) = \dfrac{\pi}{4} \qquad (1)$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\dfrac{A + B}{1 - AB}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\dfrac{5x}{1 - 6x^2}\right) = \dfrac{\pi}{4} \Rightarrow \dfrac{5x}{1 - 6x^2} = \tan\dfrac{\pi}{4}$$

$$\left[\because \tan^{-1}(\theta) = \phi \Rightarrow \theta = \tan\phi\right]$$

$$\Rightarrow \qquad \qquad \dfrac{5x}{1 - 6x^2} = 1$$

$$\Rightarrow \qquad \qquad 5x = 1 - 6x^2 \qquad (1)$$

$$\Rightarrow \qquad 6x^2 + 5x - 1 = 0$$

$$\Rightarrow \qquad 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow \qquad 6x(x + 1) - 1(x + 1) = 0 \qquad (1)$$

$$\Rightarrow \qquad (6x - 1)(x + 1) = 0$$

$$\Rightarrow \qquad 6x - 1 = 0 \ \text{ or } x + 1 = 0$$

$$\Rightarrow \qquad x = 1/6 \ \text{ or } x = -1 \qquad (1)$$

But $x = -1$ does not satisfy the given equation.

Hence, required value of x is $\dfrac{1}{6}$.

46. Find the value of the following:

$$\tan\dfrac{1}{2}\left[\sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) + \cos^{-1}\left(\dfrac{1 - y^2}{1 + y^2}\right)\right],$$

if $|x| < 1$, $y > 0$ and $xy < 1$ **Delhi 2013**

> **Firstly, use the relation**
>
> $$2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \text{ to}$$
>
> convert into $\tan^{-1}$, then use identity relation $\tan(\tan^{-1}\theta) = \theta.$

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$$

$$= \tan\left[\frac{1}{2}(2\tan^{-1} x) + \frac{1}{2}(2\tan^{-1} y)\right] \quad (2)$$

$$\left[\because 2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \tan(\tan^{-1} x + \tan^{-1} y)$$

$$= \tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \frac{x+y}{1-xy} \qquad [\because \tan(\tan^{-1}\theta) = \theta] \quad (2)$$

47. Prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

<center>Delhi 2013; All India 2011, 2008C</center>

To prove,

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$$

$$\text{LHS} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{10}}\right) + \tan^{-1}\left(\frac{1}{8}\right) \qquad (1\frac{1}{2})$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1\right]$$

$$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{72}}\right) \qquad (1\frac{1}{2})$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1\right]$$

$$= \tan^{-1}\left(\frac{56 + 9}{72 - 7}\right) = \tan^{-1}\left(\frac{65}{65}\right)$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \quad (1)$$

Hence proved.

48. Prove that

$$\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Delhi 2012;HOTS

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{\cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2} + \sin^2\dfrac{x}{2} + 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}\right)$$

$$\left[\because \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \atop \text{and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right] \quad \textbf{(1)}$$

$$= \tan^{-1}\left[\frac{\left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right)}{\left(\cos\dfrac{x}{2} + \sin\dfrac{x}{2}\right)^2}\right]$$

$$[\because a^2 - b^2 = (a-b)(a+b) \text{ and}$$
$$(a+b)^2 = a^2 + 2ab + b^2]$$

$$= \tan^{-1}\left(\frac{\cos\dfrac{x}{2} - \sin\dfrac{x}{2}}{\cos\dfrac{x}{2} + \sin\dfrac{x}{2}}\right) \quad \textbf{(1)}$$

On dividing the numerator and denominator by $\cos x/2$, we get

$$\text{LHS} = \tan^{-1}\left(\frac{\dfrac{\cos\dfrac{x}{2}}{\cos\dfrac{x}{2}} - \dfrac{\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}}}{\dfrac{\cos\dfrac{x}{2}}{\cos\dfrac{x}{2}} + \dfrac{\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}}}\right) = \tan^{-1}\left(\frac{1 - \tan\dfrac{x}{2}}{1 + \tan\dfrac{x}{2}}\right)$$

$$= \tan^{-1} \left(\frac{\tan \dfrac{\pi}{4} - \tan \dfrac{x}{2}}{1 + \tan \dfrac{\pi}{4} \cdot \tan \dfrac{x}{2}} \right) \qquad (1)$$

$$\left[\begin{array}{l} \because 1 = \tan \dfrac{\pi}{4} \text{ and} \\[2mm] 1 \cdot \tan \dfrac{x}{2} = \tan \dfrac{\pi}{4} \cdot \tan \dfrac{x}{2} \end{array} \right]$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right]$$

$$\left[\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B) \right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

$$\left[\because \tan^{-1}(\tan \theta) = \theta; \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right]$$

$$= \text{RHS} \qquad \qquad \text{(1) Hence proved.}$$

49. Prove that

$$\cos^{-1} \left(\frac{4}{5} \right) + \cos^{-1} \left(\frac{12}{13} \right) = \cos^{-1} \left(\frac{33}{65} \right).$$

All India 2012; Delhi 2010C, 2009

To prove, $\cos^{-1}\left(\dfrac{4}{5}\right) + \cos^{-1}\left(\dfrac{12}{13}\right) = \cos^{-1}\left(\dfrac{33}{65}\right)$

Let $\qquad \cos^{-1}\left(\dfrac{4}{5}\right) = x \qquad$ ...(i)

and $\qquad \cos^{-1}\left(\dfrac{12}{13}\right) = y \qquad$ ...(ii)

$\Rightarrow \qquad \cos x = \dfrac{4}{5}$ and $\cos y = \dfrac{12}{13}$ **(1)**

We know that,

$$\sin^2 x = 1 - \cos^2 x = 1 - \dfrac{16}{25} = \dfrac{9}{25}$$

$\Rightarrow \qquad \sin x = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5}$

and $\qquad \sin^2 y = 1 - \cos^2 y = 1 - \dfrac{144}{169} = \dfrac{25}{169}$

$\sin y = \sqrt{\dfrac{25}{169}} \;\Rightarrow\; = \dfrac{5}{13}$ **(1)**

Now, we know that,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$\Rightarrow \qquad \cos(x + y) = \left(\dfrac{4}{5} \times \dfrac{12}{13}\right) - \left(\dfrac{3}{5} \times \dfrac{5}{13}\right)$

$$= \dfrac{48}{65} - \dfrac{15}{65}$$

$\Rightarrow \qquad \cos(x + y) = \dfrac{33}{65}$ **(1)**

$\Rightarrow \qquad x + y = \cos^{-1}\dfrac{33}{65}$

$\Rightarrow \quad \cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65} \qquad$ ...(iii)

[From Eqs. (i), (ii) and (iii)] **(1) Hence proved.**

50. Prove that $\sin^{-1}\left(\dfrac{63}{65}\right) = \sin^{-1}\left(\dfrac{5}{13}\right) + \cos^{-1}\left(\dfrac{3}{5}\right)$.

Foreign 2012

To prove,

$$\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$\text{RHS} = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Let $\sin^{-1}\dfrac{5}{13} = x \Rightarrow \sin x = \dfrac{5}{13}$ ...(i)

and $\cos^{-1}\dfrac{3}{5} = y \Rightarrow \cos y = \dfrac{3}{5}$ ...(ii) **(1)**

Also, $\cos x = \sqrt{1 - \sin^2 x}$

$$= \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

and $\sin y = \sqrt{1 - \cos^2 y}$

$$= \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \qquad \textbf{(1)}$$

We know that,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{5}{13} \times \frac{3}{5} + \frac{12}{13} \times \frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65} \qquad \textbf{(1)}$$

$$\Rightarrow \quad x + y = \sin^{-1}\left(\frac{63}{65}\right) \qquad \text{...(iii)}$$

$$[\because \sin\theta = \phi \Rightarrow \theta = \sin^{-1}\phi]$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right) \qquad \textbf{(1)}$$

[from Eqs. (i), (ii) and (iii)]

Hence proved.

51. Solve for x,

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x), \, x \ne \frac{\pi}{2}.$$

Foreign 2012

To solve, $2 \tan^{-1}(\sin x) = \tan^{-1}(2 \sec x)$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{2 \sin x}{1 - \sin^2 x}\right) = \tan^{-1}\left(\dfrac{2}{\cos x}\right)$ (1½)

$$\left[\because\ 2 \tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right)\right.$$

$$\left. \dfrac{1}{\cos x} \right]$$

On comparing, we get

$\Rightarrow \qquad \dfrac{2 \sin x}{\cos^2 x} = \dfrac{2}{\cos x}$ (1)

$\Rightarrow \qquad\qquad \tan x = 1$ (1/2)

$\Rightarrow \quad x = \tan^{-1}(1) = \tan^{-1}\left(\tan \dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ (1)

52. Find the value of $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x - y}{x + y}\right)$.

> **?** Use the relation
> $$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

We have, $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$

$$= \tan^{-1}\left(\frac{\dfrac{x}{y} - \dfrac{x-y}{x+y}}{1 + \dfrac{x}{y}\cdot\dfrac{x-y}{x+y}}\right) \qquad (1\tfrac{1}{2})$$

$$\left[\because \tan^{-1}\theta - \tan^{-1}\phi = \tan^{-1}\left(\frac{\theta-\phi}{1+\theta\cdot\phi}\right)\right]$$

$$= \tan^{-1}\left[\frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \qquad (1\tfrac{1}{2})$$

$$= \tan^{-1}\left(\frac{x^2 + y^2}{y^2 + x^2}\right) = \tan^{-1}(1)$$

$$\left[\because \text{principal value branch of } \tan^{-1} x \text{ is } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} \qquad (1)$$

53. Prove that
$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right).$$

All India 2011; Delhi 2009C, 2008C

> **?** Use the relation, $2 \tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$
>
> and then $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$$\text{LHS} = 2 \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2 \times (1/2)}{1 - (1/2)^2}\right] + \tan^{-1}\frac{1}{7}$$

$$\left[\because\ 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] \quad (1\tfrac{1}{2})$$

$$= \tan^{-1}\left(\frac{1}{1 - \dfrac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{1}{3/4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\dfrac{4}{3} + \dfrac{1}{7}}{1 - \dfrac{4}{3} \times \dfrac{1}{7}}\right) (1\tfrac{1}{2})$$

$$\left[\because\ \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\dfrac{28+3}{21}}{\dfrac{21-4}{21}}\right) = \tan^{-1}\frac{31}{17} = \text{RHS} \qquad (1)$$

Hence proved.

54. Prove that $\dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\left(\dfrac{1}{3}\right) = \dfrac{9}{4}\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$.

Foreign 2011

Method I

Let $\sin^{-1}\left(\dfrac{1}{3}\right) = x$ and $\sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right) = y$

Then, we get

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}$$

Now, $\cos^2 x = 1 - \sin^2 x = 1 - \dfrac{1}{9} = \dfrac{8}{9}$

$\Rightarrow \quad \cos x = \sqrt{\dfrac{8}{9}} = \dfrac{2\sqrt{2}}{3}$ **(1)**

Similarly, $\cos^2 y = 1 - \sin^2 y = 1 - \dfrac{8}{9} = \dfrac{1}{9}$

$\Rightarrow \quad \cos y = \sqrt{\dfrac{1}{9}} = \dfrac{1}{3}$ **(1)**

Now, $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \frac{1}{3} \times \frac{1}{3} + \frac{2\sqrt{2}}{3} \times \frac{2\sqrt{2}}{3}$$

$$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1 \qquad \textbf{(1)}$$

$\Rightarrow \quad \sin(x + y) = 1 = \sin\dfrac{\pi}{2}$

$\Bigg[\because$ principal value branch of $\sin^{-1} x$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\Bigg]$$

$$x + y = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\left[\because x = \sin^{-1}\left(\frac{1}{3}\right) \text{ and } y = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$\Rightarrow \quad \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$$

[multiplying both sides by 9/4]

$$\Rightarrow \quad \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \quad \text{(1)}$$

Hence proved.

Method II

To prove that

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\text{LHS} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9}{4}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right] \quad \text{(1)}$$

$$= \frac{9}{4}\left[\cos^{-1}\left(\frac{1}{3}\right)\right] \quad \left[\because \cos^{-1}\theta = \frac{\pi}{2} - \sin^{-1}\theta\right]$$

$$\text{(1)}$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{1 - \frac{1}{9}}\right)$$

$$\left[\because \cos^{-1}x = \sin^{-1}\sqrt{1 - x^2}\right] \text{ (1)}$$

$$= \frac{9}{4}\sin^{-1}\left(\sqrt{\frac{8}{9}}\right) \quad \text{(1)}$$

$$= \text{RHS} \qquad \qquad \text{\textbf{Hence proved.}}$$

55. Prove that $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \dfrac{1}{2}\tan^{-1}\dfrac{4}{3}$.

To prove,

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \left(\frac{1}{2}\right)\tan^{-1}\left(\frac{4}{3}\right) \qquad \ldots(i)$$

Above equation may be rewritten as

$$2\left[\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)\right] = \tan^{-1}\left(\frac{4}{3}\right) \qquad \ldots(ii)$$

(1/2)

Now, $\text{LHS} = 2\left[\tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{2}{9}\right)\right]$

$$= 2\left[\tan^{-1}\left(\dfrac{\dfrac{1}{4}+\dfrac{2}{9}}{1-\dfrac{1}{4}\times\dfrac{2}{9}}\right)\right] \qquad (1)$$

$$\left[\because\ \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right]$$

$$= 2\tan^{-1}\left(\dfrac{\dfrac{9+8}{36}}{\dfrac{36-2}{36}}\right)=2\tan^{-1}\left(\dfrac{17}{34}\right)$$

$$= 2\tan^{-1}\left(\dfrac{1}{2}\right)=\tan^{-1}\left[\dfrac{2\times\dfrac{1}{2}}{1-\left(\dfrac{1}{2}\right)^2}\right] \qquad (1)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\dfrac{1}{1-\dfrac{1}{4}}\right)=\tan^{-1}\left(\dfrac{4}{3}\right)=\text{RHS} \qquad (1\tfrac{1}{2})$$

Hence proved.

56. Solve for x, $\cos(2\sin^{-1}x)=\dfrac{1}{9}$, $x>0$.

All India 2011C; HOTS

Given equation is

$$\cos(2\sin^{-1} x) = \frac{1}{9}, \quad x > 0 \qquad \ldots\text{(i)}$$

Put $\quad \sin^{-1} x = y \implies x = \sin y \qquad$ **(1/2)**

Then, Eq. (i) becomes, $\cos 2y = \dfrac{1}{9}$

$[\because \sin y = x]$

$\implies 1 - 2\sin^2 y = \dfrac{1}{9}$

$$[\because \cos 2\theta = 1 - 2\sin^2 \theta] \quad \textbf{(1)}$$

$\implies \quad 2\sin^2 y = 1 - \dfrac{1}{9} = \dfrac{8}{9} \qquad \textbf{(1/2)}$

$\implies \quad \sin^2 y = \dfrac{4}{9} \implies x^2 = \dfrac{4}{9} \qquad [\because \sin y = x]$

$\implies \quad x = \pm\dfrac{2}{3} \quad$ [taking square root] **(1)**

But it is given that, $x > 0$

$\therefore \quad x = \dfrac{2}{3} \qquad\qquad\qquad \textbf{(1)}$

Alternate Method

Given equation is $\cos(2\sin^{-1} x) = \dfrac{1}{9}, x > 0$

$\implies \quad \cos\left(\sin^{-1} 2x\sqrt{1-x^2}\right) = \dfrac{1}{9}$

$$\left[\because 2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})\right] \quad \textbf{(1)}$$

$\implies \cos\left[\cos^{-1}\sqrt{1-(2x\sqrt{1-x^2})^2}\right] = \dfrac{1}{9}$

$$[\because \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}]\textbf{(1)}$$

$\implies \quad \sqrt{1 - 4x^2(1-x^2)} = \dfrac{1}{9} \quad [\because \cos(\cos^{-1}\theta) = \theta]$

On squaring both sides, we get

$$81(1 - 4x^2 + 4x^4) = 1$$

$$\Rightarrow \qquad 324x^4 - 324x^2 + 80 = 0$$
$$\Rightarrow \qquad 81x^4 - 81x^2 + 20 = 0$$

[dividing both sides by 4]

$$\Rightarrow \quad 81x^4 - 45x^2 - 36x^2 + 20 = 0$$
$$\Rightarrow \quad 9x^2(9x^2 - 5) - 4(9x^2 - 5) = 0$$
$$\Rightarrow \qquad (9x^2 - 5)(9x^2 - 4) = 0$$
$$\Rightarrow \qquad x^2 = \frac{5}{9} \text{ or } \frac{4}{9} \Rightarrow x = \pm\frac{\sqrt{5}}{3} \text{ or } \pm\frac{2}{3}$$

But $\quad x > 0$

$$\therefore \quad x = +\frac{\sqrt{5}}{3} \text{ or } \frac{2}{3} \qquad\qquad (1)$$

But here, $x = \dfrac{\sqrt{5}}{3}$ does not satisfy the given equation.

$\therefore \qquad x = 2/3$ is the only solution. **(1)**

NOTE While solving an equation, please be careful on squaring the equation. Sometimes, it may give extra value, which do not satisfy the given equation.

57. Prove that $2\tan^{-1}\dfrac{3}{4} - \tan^{-1}\dfrac{17}{31} = \dfrac{\pi}{4}$.

Delhi 2011C

Firstly, apply $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ to evaluate $2\tan^{-1}\left(\dfrac{3}{4}\right)$ and then apply

$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ and get the desired result.

To prove that $2\tan^{-1}\left(\dfrac{3}{4}\right) - \tan^{-1}\left(\dfrac{17}{31}\right) = \dfrac{\pi}{4}$

$$\text{LHS} = 2 \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{2 \times \frac{3}{4}}{1 - \frac{9}{16}}\right) - \tan^{-1}\left(\frac{17}{31}\right) \qquad (1)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{3/2}{7/16}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}}\right) \qquad (1)$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{24 \times 31 - 17 \times 7}{7 \times 31 + 24 \times 17}\right)$$

$$= \tan^{-1}\left(\frac{744 - 119}{217 + 408}\right) = \tan^{-1}\left(\frac{625}{625}\right)$$

$$= \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan \frac{\pi}{4}\right) \qquad \left[\because 1 = \tan \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} = \text{RHS}$$

$$\left[\because \text{principal value branch of} \right.$$
$$\left. \tan^{-1} x \text{ is}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right).\right] \qquad (2)$$

58. Solve for x,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3},$$

$-1 < x < 1.$

> Firstly, write $\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ by applying formula $\cot^{-1} x = \tan^{-1}\frac{1}{x}$ and then proceed further.

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, \quad -1 < x < 1$$

We know that, $\cot^{-1} x = \tan^{-1}\frac{1}{x}$, so by using this result, we may write

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad (1/2)$$

Then, given equation becomes

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \qquad (1/2)$$

$$\Rightarrow \qquad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \qquad (1/2)$$

$$\Rightarrow \qquad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \qquad \frac{2x}{1-x^2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \qquad 2\sqrt{3}x = 1-x^2 \qquad (1)$$

$$\Rightarrow \qquad x^2 + 2\sqrt{3}x - 1 = 0$$

$$-2\sqrt{3} \pm \sqrt{12+4}$$

$$\Rightarrow \qquad x = \frac{\cdot}{2}$$

$$\left[\because \text{ we know that, } x = \frac{-b \pm \sqrt{D}}{2a} \text{ where,}\right.$$

$$\left. D = b^2 - 4ac\right]$$

$$\Rightarrow x = \frac{-2\sqrt{3} \pm 4}{2} = \frac{4 - 2\sqrt{3}}{2} \text{ or } \frac{-4 - 2\sqrt{3}}{2} \quad \textbf{(1)}$$

$$\Rightarrow x = 2 - \sqrt{3} \text{ or } -(2 + \sqrt{3})$$

But it is given that $-1 < x < 1$, so $x = -(2 + \sqrt{3})$ is rejected. Hence, $x = 2 - \sqrt{3}$ **(1/2)**

59. Prove that

$$\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1).$$
Delhi 2010;HOTS

> Put $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$
>
> and then use $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

To prove, $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right), x \in (0, 1)$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left[\frac{1 - (\sqrt{x})^2}{1 + (\sqrt{x})^2}\right] \quad \textbf{(1)}$$

On substituting $\sqrt{x} = \tan\theta$, we get

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \qquad (1)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2\theta) \qquad (1)$$

$$\left[\because \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A\right]$$

$$= \frac{1}{2}(2\theta) = \theta \qquad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$= \tan^{-1}\sqrt{x} \qquad [\because \theta = \tan^{-1}\sqrt{x}]\ (1)$$

$$= \text{LHS} \qquad \textbf{Hence proved.}$$

Alternate Method

To prove , $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x\in(0,1)$

$$\text{LHS} = \tan^{-1}\sqrt{x} = \frac{1}{2}(2\tan^{-1}\sqrt{x})$$

$$= \frac{1}{2}\times\cos^{-1}\left[\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right] \qquad (2)$$

$$\left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \text{RHS} \qquad (2)$$

$$\textbf{Hence proved.}$$

60. Prove that

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right). \quad \text{Delhi 2010}$$

> Use the relation $\sin^2 x + \cos^2 x = 1$. Find the value of $\sin x, \cos x, \sin y$ and $\cos y$, then use relation $\sin(x + y) = \sin x \cos y + \cos x \sin y$

Let $\cos^{-1} \dfrac{12}{13} = x$ and $\sin^{-1} \dfrac{3}{5} = y$

$\Rightarrow \quad \cos x = \dfrac{12}{13}$ and $\sin y = \dfrac{3}{5}$ **(1)**

$\therefore \quad \sin^2 x = 1 - \cos^2 x$

$$= 1 - \left(\dfrac{12}{13}\right)^2 \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$= 1 - \dfrac{144}{169} = \dfrac{25}{169} \quad \Rightarrow \quad \sin x = \sqrt{\dfrac{25}{169}} = \dfrac{5}{13}$

and $\cos^2 y = 1 - \sin^2 y = 1 - \left(\dfrac{3}{5}\right)^2 = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\Rightarrow \quad \cos y = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$ **(1)**

Now, $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\Rightarrow \sin(x + y) = \left(\dfrac{5}{13} \times \dfrac{4}{5}\right) + \left(\dfrac{12}{13} \times \dfrac{3}{5}\right)$

$$= \dfrac{20}{65} + \dfrac{36}{65} = \dfrac{56}{65} \quad \textbf{(1)}$$

$\Rightarrow \quad \sin(x + y) = \dfrac{56}{65}. \Rightarrow x + y = \sin^{-1}\left(\dfrac{56}{65}\right)$

$\Rightarrow \cos^{-1}\left(\dfrac{12}{13}\right) + \sin^{-1}\left(\dfrac{3}{5}\right) = \sin^{-1}\left(\dfrac{56}{65}\right)$

$\left[\because x = \cos^{-1}\left(\dfrac{12}{13}\right) \text{ and } y = \sin^{-1}\left(\dfrac{3}{5}\right)\right]$ **(1)**

Hence proved.

61. Prove that

$$\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

$$= 3\theta = 3\tan^{-1} x \qquad\qquad [\because \theta = \tan^{-1} x] \ \textbf{(1)}$$

$$= \tan^{-1} x + 2\tan^{-1} x$$

$$= \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad\qquad \textbf{(1)}$$

$$\left[\because 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right] = \text{LHS}$$

Hence proved.

62. Prove that $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\dfrac{1+x^2}{2+x^2}}.$

To prove $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\dfrac{1+x^2}{2+x^2}}$

$$\text{LHS} = \cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$$

Let $\cot^{-1} x = \theta \implies x = \cot\theta \qquad\qquad \textbf{(1/2)}$

Then, $\text{LHS} = \cos[\tan^{-1}(\sin\theta)]$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\operatorname{cosec}\theta}\right)\right] \qquad \textbf{(1/2)}$$

$$\left[\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+\cot^2\theta}}\right)\right]$$

$$[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta]$$

$$= \cos\left[\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right] \qquad [\because \cot\theta = x]$$

$$\textbf{(1)}$$

where, $\tan^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \phi$ or $\tan\phi = \dfrac{1}{\sqrt{1+x^2}}$

$$= \dfrac{1}{\sec\phi} \qquad \left[\because \sec\theta = \dfrac{1}{\cos\theta}\right]$$

$$= \dfrac{1}{\sqrt{1+\tan^2\phi}} \qquad [\because \tan^2\theta + 1 = \sec^2\theta]$$

$$= \dfrac{1}{\sqrt{1+\dfrac{1}{1+x^2}}} \qquad \left[\because \tan\phi = \dfrac{1}{\sqrt{1+x^2}}\right] \text{(1)}$$

$$= \dfrac{1}{\sqrt{\dfrac{1+x^2+1}{1+x^2}}} = \sqrt{\dfrac{1+x^2}{2+x^2}} = \text{RHS} \qquad \text{(1)}$$

Hence proved.

63. Solve for x, $\cos^{-1}x + \sin^{-1}\left(\dfrac{x}{2}\right) = \dfrac{\pi}{6}$.

Given, $\cos^{-1}x + \sin^{-1}\left(\dfrac{x}{2}\right) = \dfrac{\pi}{6}$

$$\Rightarrow \quad \cos^{-1}x = \dfrac{\pi}{6} - \sin^{-1}\dfrac{x}{2}$$

$$\Rightarrow \quad x = \cos\left(\dfrac{\pi}{6} - \sin^{-1}\dfrac{x}{2}\right)$$

$$\Rightarrow \quad x = \cos\dfrac{\pi}{6}\cos\left(\sin^{-1}\dfrac{x}{2}\right)$$

$$+ \sin\dfrac{\pi}{6}\sin\left(\sin^{-1}\dfrac{x}{2}\right) \text{(1)}$$

$$[\because \cos(x-y) = \cos x \cos y + \sin x \sin y]$$

$$\Rightarrow \quad x = \dfrac{\sqrt{3}}{2}\cos\left(\sin^{-1}\dfrac{x}{2}\right) + \dfrac{1}{2}\cdot\dfrac{x}{2}$$

$$[\because \sin(\sin^{-1}\theta) = \theta]$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\cos\left(\cos^{-1}\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$

$$\left[\because \; \sin^{-1}\theta = \cos^{-1}\sqrt{1-\theta^2}\right]$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) + \frac{x}{4}$$

$$\Rightarrow \quad x - \frac{x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right)$$

$$\Rightarrow \quad \frac{3x}{4} = \frac{\sqrt{3}}{2}\left(\sqrt{1-\frac{x^2}{4}}\right) \qquad (1)$$

On squaring both sides, we get

$$\frac{9x^2}{16} = \frac{3}{4}\left(1-\frac{x^2}{4}\right) \quad \Rightarrow \quad \frac{3}{4}x^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow \quad \frac{3}{4}x^2 + \frac{x^2}{4} = 1 \quad \Rightarrow \quad \frac{4x^2}{4} = 1$$

$$\Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \qquad (1)$$

But $x = -1$, do not satisfy the given equation.

Hence, $x = 1$. $\qquad (1)$

64. Prove that $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$.

<div align="right">All India 2010C</div>

To prove $\; 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

LHS $= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$ $\qquad$ ...(i)

We know that, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Using this identity, we can write

$$2 \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left[\frac{2 \times \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2}\right] = \tan^{-1}\left(\frac{2/3}{1 - \frac{1}{9}}\right)$$

$$\Rightarrow \quad 2 \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right) \qquad (1\frac{1}{2})$$

On putting the value of $2 \tan^{-1}\left(\frac{1}{3}\right)$ in Eq.(i),

we get

$$\text{LHS} = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{21 + 4}{28}}{\frac{28 - 3}{28}}\right)$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}(1) \qquad (1\frac{1}{2})$$

$\because$ The principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \text{LHS} = \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \qquad (1)$$

Hence proved.

65. Solve for x,

$$\tan^{-1} \frac{x}{2} + \tan^{-1} \frac{x}{3} = \frac{\pi}{4}, \sqrt{6} > x > 0. \text{ Delhi 2010C}$$

Given, $\tan^{-1}\dfrac{x}{2} + \tan^{-1}\dfrac{x}{3} = \dfrac{\pi}{4}$, $\sqrt{6} > x > 0$

$$\Rightarrow \quad \tan^{-1}\left(\dfrac{\dfrac{x}{2}+\dfrac{x}{3}}{1-\dfrac{x^2}{6}}\right) = \dfrac{\pi}{4}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right] \quad \text{(1½)}$$

$$\Rightarrow \dfrac{\dfrac{3x+2x}{6}}{\dfrac{6-x^2}{6}} = \tan\dfrac{\pi}{4} \Rightarrow \dfrac{5x}{6-x^2} = 1 \left[\because \tan\dfrac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad 5x = 6 - x^2 \Rightarrow x^2 + 5x - 6 = 0$$

$$\Rightarrow \quad x^2 + 6x - x - 6 = 0$$

$$\Rightarrow \quad x(x+6) - 1(x+6) = 0$$

$$\Rightarrow \quad (x-1)(x+6) = 0$$

$$\Rightarrow \quad x = 1 \ \text{ or } \ -6 \quad \text{(1½)}$$

But it is given that, $\sqrt{6} > x > 0 \Rightarrow x > 0$

$\therefore x = -6$ is rejected. Hence, $x = 1$ \quad **(1)**

66. Solve for x,

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\dfrac{8}{79}\right), x>0.$$

<div align="right">Delhi 2010C</div>

Apply $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ in LHS of given equation and then proceed further to obtain the desired result.

Given equation is

$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\dfrac{8}{79}\right), x > 0$

$\Rightarrow \tan^{-1}\left[\dfrac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right] = \tan^{-1}\left(\dfrac{8}{79}\right)$ (1½)

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right]$

$\Rightarrow \left[\dfrac{2x}{1-(x^2-4)}\right] = \dfrac{8}{79}$ (1/2)

$[\because (a+b)(a-b) = a^2 - b^2]$

$\Rightarrow \dfrac{2x}{5-x^2} = \dfrac{8}{79} \Rightarrow \dfrac{x}{5-x^2} = \dfrac{4}{79}$

$\Rightarrow \qquad\qquad 79x = 20 - 4x^2$

$\Rightarrow \qquad 4x^2 + 79x - 20 = 0$

$\Rightarrow \qquad 4x^2 + 80x - x - 20 = 0$

$\Rightarrow \qquad 4x(x+20) - 1(x+20) = 0$

$\Rightarrow \qquad (4x-1)(x+20) = 0$

$\Rightarrow \qquad\qquad x = \dfrac{1}{4} \text{ or } -20$ (1)

But it is given that, $x > 0$

$\therefore \qquad\qquad x = -20$ is rejected.

Hence, $\quad x = 1/4$ (1)

67. Prove that

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}.$$

To prove $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$

$\text{LHS} = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$

(1)

$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{\frac{144}{169}} + \frac{5}{13} \times \sqrt{\frac{9}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$

$$= \sin^{-1}\left[\left(\frac{4}{5} \times \frac{12}{13}\right) + \left(\frac{5}{13} \times \frac{3}{5}\right)\right] + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} + \frac{15}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right) \qquad (1)$$

$$= \sin^{-1}\left[\frac{63}{65}\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right]$$

$$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{\frac{4225-256}{4225}} + \frac{16}{65}\sqrt{\frac{4225-3969}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65} \times \sqrt{\frac{3969}{4225}} + \frac{16}{65} \times \sqrt{\frac{256}{4225}}\right)$$

$$= \sin^{-1}\left(\frac{63}{65} \times \frac{63}{65} + \frac{16}{65} \times \frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{3969+256}{4225}\right) = \sin^{-1}\left(\frac{4225}{4225}\right) = \sin^{-1}(1)$$

$\because$ The principal value branch of $\sin^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \text{LHS} = \sin^{-1}\left(\sin\frac{\pi}{2}\right) = \frac{\pi}{2} = \text{RHS} \qquad (1)$$

Hence proved.

Alternate Method Given equation,

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2} \quad \text{...(i)}$$

$$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{16}{65}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right) \quad \text{...(ii)}$$

$$\left[\because \frac{\pi}{2} - \sin^{-1}\theta = \cos^{-1}\theta\right] \text{ (1)}$$

Hence, Eqs. (i) and (ii) are equivalent. Now, we have to prove Eq. (ii).

Now do same as Que 42.

68. Prove that
$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}.$$

<div align="right">All India 2009C</div>

. To prove, $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

$$\text{LHS} = \left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{9}{20}}\right) - \tan^{-1}\frac{8}{19}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \text{(1)}$$

$$= \tan^{-1}\left(\frac{27/20}{11/20}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right) \qquad \text{(1/2)}$$

$$= \tan^{-1}\left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \times \frac{8}{19}}\right) \qquad\qquad \text{(1)}$$

$$= \tan^{-1}\left(\frac{\frac{513-88}{209}}{\frac{209+216}{209}}\right) = \tan^{-1}\left(\frac{425}{209} \times \frac{209}{425}\right) \text{(1/2)}$$

$$= \tan^{-1}(1)$$

$\because$ The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \quad \text{LHS} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} \qquad \text{(1)}$$

$$= \text{RHS} \qquad\qquad \textbf{Hence proved.}$$

69. Prove that

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \frac{\pi}{4}.$$ **All India 2009C; Delhi 2008, 2008C**

> ? Apply the identity,
>
> $$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ in first two}$$
>
> terms and the last two terms of LHS and then apply the same identity again to the get the RHS.

To prove $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$+ \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$$

$$\text{LHS} = \left[\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right)\right]$$

$$+ \left[\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right]$$

On applying the identity

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we get}$$

$$\text{LHS} = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right) + \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right) \quad (1\tfrac{1}{2})$$

$$= \tan^{-1}\left(\frac{8/15}{14/15}\right) + \tan^{-1}\left(\frac{15/56}{55/56}\right)$$

$$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$$

•

$$= \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}\right) = \tan^{-1}\left(\frac{\frac{44+21}{77}}{\frac{77-12}{77}}\right)(1)$$

$$= \tan^{-1}\left[\frac{65/77}{65/77}\right] = \tan^{-1}(1) \qquad (1)$$

$\because$ The principal value branch of $\tan^{-1}x$ is

$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$$\therefore \text{LHS} = \tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right] = \frac{\pi}{4} = \text{RHS} \qquad (1/2)$$

70. Solve for x,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}. \text{ Delhi 2009C}$$

Do same as Que 38. **[Ans. $\pm 1/2$]**

71. Solve for x,

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1}x, 0 < x < 1.$$
$$\text{Delhi 2008C}$$

The given equation is

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4} \qquad (1/2)$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\dfrac{1+x}{1-x} - x}{1 + \left(\dfrac{1+x}{1-x}\right)\cdot x}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right] \; (1½)$$

$$\Rightarrow \quad \frac{1+x-x+x^2}{1-x+x+x^2} = \tan\frac{\pi}{4} \qquad (1)$$

$$\Rightarrow \quad \frac{1+x^2}{1+x^2} = 1 \Rightarrow 1 = 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

Hence, the given equation has many solutions. (1)